

Verification of the semiclassical method for an electron moving in a homogeneous magnetic field

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(Received 14 March 2001; published 24 September 2001)

A procedure based on the semiclassical approximation for high energy levels is developed to yield solutions to the classical equation of charge motion and to the Bargmann-Michel-Telegdi spin equation. To this end, exact solutions to the Klein-Gordon and the Dirac-Pauli equations are used. The essence of the procedure under review is that the quantum state of a charged particle in a homogeneous magnetic field is represented as a superposition of states corresponding to the neighboring energy levels. As a consequence, the expectation values of the momentum and spin operators with respect to the resulting nonstationary wave function (packet) strictly obey the classical equations of charge motion and spin precession.

DOI: 10.1103/PhysRevE.64.046503

PACS number(s): 41.60.Ap, 03.65.Sq, 03.50.De

I. INTRODUCTION

In the semiclassical approach, the charge motion is described in a classical way, and quantum transitions are calculated on the basis of matrix elements [1]. This method is known to be applicable in the ultrarelativistic case at ultrahigh electron energies. For example, at energies of ~ 2.5 GeV, which are typical for current storage rings with a characteristic parameter $H\rho \sim 10^7$ Oe cm, the electron energy levels amount to $n \sim 10^{17}$. For radiative transitions to lower energy levels, $\Delta n/n \sim 10^{-7}$. In this case, according to the uncertainty principle, all quantum processes occur in a region as small as a few angstroms (see [2] for details). Clearly, this is a negligible value for an orbital radius of about 10 m. Hence, the classical, or more precisely, semiclassical concept of the electron trajectory can be used here.

In the theory of synchrotron radiation, the semiclassical method allows all quantum corrections to the synchrotron radiation power to be calculated, including the spin-flip effects [3,4]. This method is straightforward to use. What is more, it provides a visual picture of physical processes occurring at ultrahigh electron energies (recoil effects, radiative spin self-polarization, spin magnetic-moment radiation, mixed charge and magneton radiation, radiation associated with the anomalous part of the electron magnetic moment, etc.).

Impressive success has been achieved in the semiclassical synchrotron radiation theory. However, to our knowledge, rigorous mathematical substantiation of this method based on exact solutions to the Klein-Gordon equation or to the Dirac-Pauli equation is lacking. General statements of the theory were, in fact, postulated on the basis of the uncertainty principle (see [1]).

In this paper, solutions to the classical equation of charge motion and to the Bargmann-Michel-Telegdi (BMT) spin precession equation [5] are constructed in the framework of quantum mechanics, using nonstationary wave functions (see the general statements in [6]). The latter are, in their turn, exact solutions to the Klein-Gordon equation or to the Dirac-

Pauli equation for a charged particle moving in a homogeneous magnetic field. A simplified version of this method was used in [7] to reveal a physical pattern underlying the behavior of the longitudinal electron-spin polarization in a magnetic field. This approach is adopted in [3] to describe quantum neutron spin-flip transitions.

Our task is to explore the time evolution of the expectation values of the momentum and spin operators for a charged particle moving in a homogeneous magnetic field. To demonstrate the capability of the method discussed, let us examine the problem at hand first. In Sec. II, we will consider a scalar particle and compare the time evolution of the expectation value of the momentum operator for this particle with the solution to the classical equation of charge motion. These are found to differ by a factor. Section III will focus on a spin- $\frac{1}{2}$ particle. In addition to evolution of the expectation value of the momentum operator, we will discuss the behavior of the expectation value of the spin operator. The latter also differs from the solution to the BMT equation by the above factor only. Finally, in Sec. IV, we will introduce a procedure that will help to eliminate the difference between the quantum and classical approaches. To this end, the state of a particle will be represented by a wave packet so that the time evolution of expectation values of the corresponding operators with respect to this nonstationary wave function will coincide with that predicted by the classical mechanics.

II. MOTION OF A SCALAR PARTICLE

Let us assume that the state of a scalar particle is formed as a wave packet by superposing exact solutions to the Klein-Gordon equation corresponding to three neighboring energy levels $n-1$, n , and $n+1$:

$$\Psi(\mathbf{r}, t) = \sum_{m=n-1}^{n+1} A_m \psi_m(\mathbf{r}) \exp\left(-\frac{i}{\hbar} m_0 c^2 B_m t\right),$$

where $\psi_m(\mathbf{r})$ are the stationary solutions to the Klein-Gordon equation in a homogeneous magnetic field (see Appendix A) and A_m are the expansion coefficients.

To determine the coefficients A_m in the semiclassical approximation ($n \gg 1$), the probability that a particle will be found at each of the levels is assumed to be the same. For

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$n \gg 1$, this is a natural assumption. Normalizing the wave function $\Psi(\mathbf{r}, t)$ to unit probability yields a relation of the form

$$\sum_{m=n-1}^{n+1} A_m^* A_m = 1.$$

In what follows the explicit form of A_m is immaterial but different relations between these coefficients are of great importance. In particular, we need the following equation:

$$\mathfrak{A}_{\text{KG}} = A_n^* A_{n-1} + A_{n+1}^* A_n = \frac{2}{3}.$$

Now let us calculate the expectation value of the momentum operator $\hat{\mathbf{P}}$ in the state $\Psi(\mathbf{r}, t)$. In the first step, we calculate matrix elements of the $\langle \psi_{m'} | \hat{\mathbf{P}} | \psi_m \rangle$ type with respect to stationary solutions to the Klein-Gordon equation. Then, using the semiclassical condition ($n \gg 1$) for each matrix element [see Eqs. (A2)] we write $\langle \Psi | \hat{\mathbf{P}} | \Psi \rangle_t$ explicitly to give

$$\begin{aligned} \langle \hat{P}_x \rangle_t &= \frac{i}{2} m_0 c b_{\perp} [\mathfrak{A}_{\text{KG}} \exp(i\omega t) - \mathfrak{A}_{\text{KG}}^* \exp(-i\omega t)] \\ &= -\frac{2}{3} m_0 c b_{\perp} \sin \omega t, \end{aligned}$$

$$\begin{aligned} \langle \hat{P}_y \rangle_t &= \frac{1}{2} m_0 c b_{\perp} [\mathfrak{A}_{\text{KG}} \exp(i\omega t) + \mathfrak{A}_{\text{KG}}^* \exp(-i\omega t)] \\ &= \frac{2}{3} m_0 c b_{\perp} \cos \omega t, \end{aligned}$$

$$\langle \hat{P}_z \rangle_t = m_0 c b_z.$$

Here

$$\omega = \frac{m_0 c^2}{\hbar} (B_{n+1} - B_n) = \frac{e_0 H}{m_0 c^2 B_n} \rightarrow \frac{e_0 H}{m_0 c^2 \gamma}$$

is the frequency that, in essence, coincides with the cyclotron frequency of rotation of a classical particle in a plane perpendicular to the magnetic field vector. In this approximation, $B_n \rightarrow \gamma$, where $\gamma = (1 - \beta^2)^{-1/2}$ is the Lorentz factor and β is a velocity of the particle in classical theory.

This leads us to conclude that the expectation values differ from the corresponding classical solutions to the equation of motion of a charged particle only by the factor $2/3$ in the $\langle \hat{P}_x \rangle_t$ and $\langle \hat{P}_y \rangle_t$ components. Note that the same result was also obtained in [8] for an ordinary one-dimensional quantum harmonic oscillator. In the discussion below, we will show that this factor can be made as large as unity.

III. CHARGE MOTION AND SPIN PRECESSION OF A DIRAC-PAULI PARTICLE

When considering the Dirac-Pauli particle the spin polarization of the particle must be taken into account. A super-

position of exact solutions to the Dirac-Pauli equation corresponding to the three neighboring energy levels $n-1$, n , and $n+1$ has the form

$$\Psi(\mathbf{r}, t) = \sum_{\zeta} \sum_{m=n-1}^{n+1} A_{\zeta m} \psi_{m\zeta}(\mathbf{r}) \exp\left(-\frac{i}{\hbar} m_0 c^2 B_m \zeta t\right), \quad (1)$$

where $\psi_m(\mathbf{r})$ are the stationary solutions to the Dirac-Pauli equation (see Appendix B) and $A_{\zeta m}$ are expansion coefficients. Note that, in this case, allowance is made for polarization states defined by the quantum number $\zeta = \pm 1$. Let us assume that a given longitudinal-polarization state at time $t = 0$ for an electron located at level m is of the form¹

$$(\boldsymbol{\sigma} \cdot \hat{\mathbf{P}}) \sum_{\zeta} A_{\zeta m} \psi_{m\zeta} = \lambda_m A_{\zeta m} \psi_{m\zeta}. \quad (2)$$

Having solved Eq. (2) we find the eigenvalue

$$\lambda_m = \varepsilon \sqrt{b_{\perp}^2 + b_z^2}, \quad \varepsilon = \pm 1.$$

In the semiclassical approximation, it can be supposed that for all values of m the spin coefficients are the same, i.e., $C_i(m, \zeta) \approx C_i(n, \zeta)$ and, moreover, $\lambda_m \approx \lambda_n$ [see Eqs. (B3) and (B4)]. Then we have

$$A_{1m} = \kappa A_{-1m}, \quad \kappa = \frac{b_z + \varepsilon b \sqrt{b_{\perp}^2 + b_z^2}}{B_{n\zeta} b_{\perp}}.$$

Let us also require that the wave function $\Psi(\mathbf{r}, t)$ be normalized to unit probability. As a result, we get

$$\sum_{\zeta} \sum_{m=n-1}^{n+1} A_{\zeta m}^* A_{\zeta m} = 1. \quad (3)$$

In what follows we need the relations

$$\mathfrak{A}_1 = \sum_{m=n-1}^n (A_{1m}^* A_{1m+1} + A_{-1m}^* A_{-1m+1}) = \frac{2}{3}, \quad (4a)$$

$$\mathfrak{A}_2 = \sum_{m=n-1}^n A_{1m}^* A_{-1m+1} = \sum_{m=n-1}^n A_{-1m}^* A_{1m+1} = \frac{2}{3} \frac{\kappa}{\kappa+1}, \quad (4b)$$

$$\mathfrak{A}_3 = \sum_{m=n-1}^{n+1} A_{1m}^* A_{-1m} = \frac{\kappa}{\kappa^2+1}, \quad (4c)$$

$$\mathfrak{A}_4 = \sum_{m=n-1}^{n+1} (A_{1m}^* A_{1m} - A_{-1m}^* A_{-1m}) = \frac{\kappa^2 - 1}{\kappa^2 + 1}. \quad (4d)$$

To simplify the calculations we suppose that, as an electron makes a transition from one level to another, the projection of the momentum of the electron onto the direction of the

¹It is known [7] that, with regard to the anomalous magnetic moment of an electron, the operator $(\boldsymbol{\sigma} \cdot \hat{\mathbf{P}})$ is not an integral of motion.

magnetic field is conserved. In addition, random deviations of the orbital center of the electron due to radiation are neglected. These restrictions imply that $b'_z = b_z$ and $s' = s$ (see Appendix B).

Now, by analogy to a spinless particle, we find the expectation value of the operator $\hat{\mathbf{P}}$ using the state of the Dirac-Pauli particle given by Eq. (1) (see the corresponding matrix elements in Appendix B). As a result of straightforward but cumbersome calculations, we derive, in view of these restrictions, the following equations:

$$\begin{aligned} \langle \hat{P}_x \rangle_t &= \frac{i}{2} m_0 c b_\perp [\mathfrak{A}_1 \exp(i\omega t) - \mathfrak{A}_1^* \exp(-i\omega t)] \\ &= -\frac{2}{3} m_0 c b_\perp \sin \omega t, \end{aligned} \quad (5a)$$

$$\begin{aligned} \langle \hat{P}_y \rangle_t &= \frac{1}{2} m_0 c b_\perp [\mathfrak{A}_1 \exp(i\omega t) + \mathfrak{A}_1^* \exp(-i\omega t)] \\ &= \frac{2}{3} m_0 c b_\perp \cos \omega t, \end{aligned} \quad (5b)$$

$$\langle P_z \rangle_t = m_0 c b_z. \quad (5c)$$

Here

$$\omega = \frac{m_0 c^2}{\hbar} (B_{n+1\zeta} - B_{n\zeta}) = \frac{e_0 H}{m_0 c^2 B_{n\zeta}} \rightarrow \frac{e_0 H}{m_0 c^2 \gamma}$$

is the frequency that in the BMT approximation (the charge motion is independent of the spin precession) coincides with the cyclotron frequency of rotation of an electron in the plane perpendicular to the magnetic field vector. In this approximation, $B_{n\zeta} \rightarrow \gamma$.

A similar procedure applies to calculations of the expectation value of the spin operator \hat{S}^μ (see [9]) to yield

$$\hat{S}^\mu = \left(\frac{1}{m_0 c} (\boldsymbol{\sigma} \cdot \hat{\mathbf{P}}), \rho_3 \boldsymbol{\sigma} + \frac{1}{m_0 c} \rho_1 \hat{\mathbf{P}} + \frac{\mu_a}{m_0 c^2} \rho_3 \mathbf{H} \right).$$

Taking into account Eqs. (4), we obtain

$$\langle \hat{S}_x \rangle_t = -\frac{2}{3} \zeta_\perp (\cos \omega t \sin \Omega_a t + b \sin \omega t \cos \Omega_a t), \quad (6a)$$

$$\langle \hat{S}_y \rangle_t = -\frac{2}{3} \zeta_\perp (\sin \omega t \sin \Omega_a t - b \cos \omega t \cos \Omega_a t), \quad (6b)$$

$$\langle \hat{S}_z \rangle_t = \frac{B_{n\zeta}}{b} \zeta_z + \frac{b_z b_\perp}{b} \zeta_\perp \cos \Omega_a t, \quad (6c)$$

$$\langle \hat{S}^0 \rangle_t = \frac{b_z}{b} \zeta_z + B_{n\zeta} \frac{b_\perp}{b} \zeta_\perp \cos \Omega_a t. \quad (6d)$$

Here $\zeta_z^2 = 1 - \zeta_\perp^2$, $\zeta_\perp = 2\kappa(\kappa^2 + 1)^{-1}$ are constants that, as will be seen later, are analogous to some constants in the classical spin theory. Thus, as in the classical approach, the

total precession is determined not only by the cyclotron frequency ω but also by the anomalous frequency to give

$$\begin{aligned} \Omega_a &= \frac{m_0 c^2}{\hbar} (B_{n\zeta} - B_{n\zeta'}) = 2 \frac{\mu_a H}{\hbar} \frac{b}{\sqrt{b_z^2 + b^2}} \\ &\rightarrow \frac{\alpha}{2\pi} \frac{e_0 H}{m_0 c} \sqrt{1 - \beta_z^2}. \end{aligned}$$

Note that in the BMT approximation, the term comprising the anomalous magnetic moment in the spin operator \hat{S}^μ is immaterial for the result (6).

We see that, by analogy with the spinless charged particle, the results obtained in Eqs. (5a), (5b), (6a), and (6b) differ from the corresponding solutions to the classical equations of charge motion and spin precession by the factor 2/3.

IV. SEMICLASSICAL CORRESPONDENCE PRINCIPLE

To construct a more precise semiclassical theory of charge motion and spin precession, we will consider a wave packet involving exact solutions to the Dirac-Pauli equation (B2) corresponding to closely spaced energy levels.² In this case, wave function (1) can be written in the following form:

$$\Psi(\mathbf{r}, t) = \sum_{\zeta} \sum_{m \in \mathfrak{R}} A_{\zeta m} \psi_{m\zeta}(\mathbf{r}) \exp\left(-\frac{i}{\hbar} m_0 c^2 B_{m\zeta} t\right), \quad (7)$$

where \mathfrak{R} is the set of values of the principal quantum number. Let us assume that each value of m in Eq. (7) belonging to the set is much larger than unity, i.e., $m \gg 1$. At the same time, we have

$$n \gg N = m_{\max} - m_{\min} \gg 1,$$

where m_{\max} and m_{\min} are maximum and minimum values belonging to \mathfrak{R} , and n is the quantum number from \mathfrak{R} defining some energy level to which a given classical trajectory corresponds. Let us further assume that the other energy levels from the set are symmetric about level n .

The next natural step in our method is to derive the coefficients $A_{\zeta m}$. The central idea of the derivation is the same as in Eq. (3), i.e.,

$$A_{1m} = \kappa A_{-1m}, \quad \sum_{\zeta} \sum_{m \in \mathfrak{R}} A_{\zeta m}^* A_{\zeta m} = 1.$$

Then, relations (4) can easily be extended to the case under consideration, namely,

$$\mathfrak{A}_1 = \sum_{m=m_{\min}}^{m_{\max}-1} (A_{1m}^* A_{1m+1} + A_{-1m}^* A_{-1m+1}) = \frac{N-1}{N}, \quad (8a)$$

²It can be shown that there is a simpler form of this construction for the motion of a spinless charged particle.

$$\begin{aligned} \mathfrak{A}_2 &= \sum_{m=m_{\min}}^{m_{\max}-1} A_{1m}^* A_{-1m+1} = \sum_{m=m_{\min}}^{m_{\max}-1} A_{-1m}^* A_{1m+1} \\ &= \frac{N-1}{N} \frac{\kappa}{\kappa+1}, \end{aligned} \quad (8b)$$

$$\mathfrak{A}_3 = \sum_{m \in \mathfrak{R}} A_{1m}^* A_{-1m} = \frac{\kappa}{\kappa^2+1}, \quad (8c)$$

$$\mathfrak{A}_4 = \sum_{m \in \mathfrak{R}} (A_{1m}^* A_{1m} - A_{-1m}^* A_{-1m}) = \frac{\kappa^2-1}{\kappa^2+1}. \quad (8d)$$

Following the above procedure of calculating the expectation values of the momentum and spin operators and using Eqs. (8) we obtain

$$\langle \hat{P}_x \rangle_t = -\frac{N-1}{N} m_0 c b_{\perp} \sin \omega t,$$

$$\langle \hat{P}_y \rangle_t = \frac{N-1}{N} m_0 c b_{\perp} \cos \omega t,$$

$$\langle \hat{P}_z \rangle_t = m_0 c b_z,$$

$$\langle \hat{S}_x \rangle_t = -\frac{N-1}{N} \zeta_{\perp} (\cos \omega t \sin \Omega_a t + b \sin \omega t \cos \Omega_a t),$$

$$\langle \hat{S}_y \rangle_t = -\frac{N-1}{N} \zeta_{\perp} (\sin \omega t \sin \Omega_a t - b \cos \omega t \cos \Omega_a t),$$

$$\langle \hat{S}_z \rangle_t = \frac{B_n \zeta}{b} \zeta_z + \frac{b_z b_{\perp}}{b} \zeta_{\perp} \cos \Omega_a t,$$

$$\langle \hat{S}^0 \rangle_t = \frac{b_z}{b} \zeta_z + B_n \zeta \frac{b_{\perp}}{b} \zeta_{\perp} \cos \Omega_a t.$$

Thus, for $N \gg 1$, the difference of $\langle \hat{P}_x \rangle_t$, $\langle \hat{P}_y \rangle_t$, $\langle \hat{S}_x \rangle_t$, and $\langle \hat{S}_y \rangle_t$ from the results obtained by the classical theory is eliminated, and the time evolution of the expectation values $\langle \hat{\mathbf{P}} \rangle_t$ and $\langle \hat{S}^{\mu} \rangle_t$ is made to coincide with the solutions to the corresponding classical equations [10].

It is easy to see that the anomalous magnetic moment of an electron affects not only the time behavior of the spin projection onto the direction of motion (longitudinal polarization) of the particle [7] but also the behavior of spin in the plane perpendicular to the magnetic field vector, i.e., the time evolution of $\langle \hat{S}_x \rangle_t$ and $\langle \hat{S}_y \rangle_t$.

V. CONCLUDING REMARKS

It should be noted that $\langle \hat{S}^{\mu} \rangle_t$ derived in Sec. IV satisfy the relations given by the classical spin theory:

$$\langle \hat{S}^{\mu} \rangle_t \langle \hat{P}_{\mu} \rangle_t = 0, \quad \langle \hat{S}^{\mu} \rangle_t \langle \hat{S}_{\mu} \rangle_t = 1.$$

Moreover, the procedure under review can be extended to the tensor operator [9]

$$\hat{\Pi}^{\mu\nu} = (\hat{\Phi}, \hat{\Pi}),$$

$$\hat{\Phi} = -\frac{1}{m_0 c} \rho_3 (\boldsymbol{\sigma} \times \hat{\mathbf{P}}) + \frac{\mu_a}{m_0 c^2} \rho_1 (\boldsymbol{\sigma} \times \mathbf{H}),$$

$$\hat{\Pi} = \boldsymbol{\sigma} + \frac{1}{m_0 c} \rho_2 (\boldsymbol{\sigma} \times \hat{\mathbf{P}}) + \frac{\mu_a}{m_0 c^2} \mathbf{H}.$$

In the BMT approximation, the tensor operator is related to \hat{S}^{μ} by the formula [2]

$$\hat{\Pi}^{\mu\nu} = \frac{1}{m_0 c} \varepsilon^{\mu\nu\alpha\beta} \hat{S}_{\alpha} \hat{P}_{\beta}.$$

Thus, the expectation values $\langle \hat{S}^{\mu} \rangle_t$, $\langle \hat{\mathbf{P}} \rangle_t$, and $\langle \hat{\Pi}^{\mu\nu} \rangle_t$, calculated following the procedure discussed in Sec. IV, obey the classical equation of charge motion and spin precession. The approach introduced in this work substantiates the use of the semiclassical method for gaining an insight into physical phenomena associated with high-energy charged particles. In particular, the semiclassical approach can be the basis for a classical model for certain purely quantum processes. For example, it can be used to establish a relationship between spin-flip transitions and classical spin precession [3].

ACKNOWLEDGMENTS

The authors would like to thank Professor V. G. Bagrov for support and valuable discussions.

APPENDIX A

The motion of a scalar particle in a constant homogeneous magnetic field $\mathbf{H} = (0, 0, H)$ is given by the Klein-Gordon equation

$$\left(\frac{1}{c^2} \hat{E}^2 - \hat{\mathbf{P}}^2 + m_0^2 c^2 \right) \psi(\mathbf{r}, t) = 0,$$

where $\hat{E} = i\hbar \partial / \partial t$ is the energy operator, $\hat{\mathbf{P}} = \hat{\mathbf{p}} - (e/c) \mathbf{A}$ is the kinetic momentum operator, $\mathbf{A} = (-Hy/2, Hx/2, 0)$, $e = -e_0 < 0$ is the charge, and m_0 is the rest mass of the particle.

It is known that in this case the wave function $\psi(\mathbf{r}, t)$ is to be the eigenfunction of the energy operator \hat{E} , and of the projection of the kinetic momentum operator $\hat{\mathbf{P}}$ and orbital angular momentum operator $\hat{\mathbf{L}} = \mathbf{r} \times \hat{\mathbf{p}}$ onto the direction of the magnetic field vector, i.e.,

$$\hat{E} \psi = E_n \psi, \quad \hat{P}_z \psi = p_z \psi, \quad \hat{L}_z \psi = \hbar l \psi.$$

Then, in the cylindrical coordinates (r, φ, z) most suitable in this case, the solution to the Klein-Gordon equation has the form [11]

$$\psi(\mathbf{r}, t) = \frac{\sqrt{e_0 H}}{\sqrt{L} \sqrt{2\pi\hbar c}} e^{-iE_n t/\hbar} e^{ip_z z/\hbar} e^{i\varphi} I_{n,s}(\rho). \quad (\text{A1})$$

The function $I_{n,s}(\rho)$ in Eq. (A1) is defined by the Laguerre polynomials $Q_s^n(\rho)$ with the help of the following relation:

$$I_{n,s}(\rho) = \frac{1}{\sqrt{n!s!}} e^{-\rho/2} Q_s^{n-s} \rho^{(n-s)/2}, \quad \rho = \frac{e_0 H}{2\hbar c} r^2.$$

Here $n = l + s = 0, 1, 2, \dots$ is the principal quantum number, and $s = 0, 1, 2, \dots$ and $l = 0, \pm 1, \pm 2, \dots$ are the radial and azimuthal quantum numbers.

The energy of the particle is written as

$$E_n = m_0 c^2 B_n = m_0 c^2 \sqrt{1 + b_z^2 + b_\perp^2},$$

$$b_\perp = 2 \sqrt{\frac{\mu_0 H}{m_0 c^2} \left(n + \frac{1}{2} \right)},$$

where $b_z = p_z/m_0 c$ is the projection of the momentum onto the direction of the field and $\mu_0 = e_0 \hbar/2m_0 c$ is the Bohr magneton.

The matrix elements of the momentum operator of the scalar particle have the form

$$\langle \psi_{m'} | \hat{P}_x | \psi_m \rangle = \frac{i}{2} m_0 c b_\perp (\delta_{m'-m, +1} - \delta_{m'-m, -1}), \quad (\text{A2a})$$

$$\langle \psi_{m'} | \hat{P}_y | \psi_m \rangle = \frac{1}{2} m_0 c b_\perp (\delta_{m'-m, +1} + \delta_{m'-m, -1}), \quad (\text{A2b})$$

$$\langle \psi_{m'} | \hat{P}_z | \psi_m \rangle = m_0 c b_z \delta_{m'm}. \quad (\text{A2c})$$

APPENDIX B

We use the solutions to the following Dirac-Pauli equation:

$$i\hbar \frac{\partial \psi}{\partial t} = \hat{H} \psi, \quad \hat{H} = c(\boldsymbol{\alpha} \cdot \hat{\mathbf{P}}) + \rho_3 m_0 c^2 + \mu_a \rho_3 (\boldsymbol{\sigma} \cdot \mathbf{H}),$$

where $\hat{\mathbf{P}} = \hat{\mathbf{p}} - (e/c)\mathbf{A}$ is the kinetic momentum operator, $\mathbf{A} = (-Hy/2, Hx/2, 0)$, \mathbf{H} is the external magnetic field, $\mu_a = (\alpha/2\pi)\mu_0$ is the anomalous part of the electron magnetic moment, and $e = -e_0 < 0$ and m_0 are the charge and rest mass of the electron.

It is known [7] that in this case a complete set of commuting operators characterizing the quantum state of the particle consists of the Hamiltonian \hat{H} , projections of the momentum operator $\hat{\mathbf{P}}$ and total angular momentum operator $\hat{\mathbf{J}} = \hat{\mathbf{L}} + \frac{1}{2}\hbar \boldsymbol{\sigma}$ onto the direction of the magnetic field vector, and the polarization operator. For definiteness, the magnetic field vector is oriented along the Z axis. According to [7] the operator

$$\hat{\Pi}_z = \sigma_z + \frac{1}{m_0 c} \rho_2 (\boldsymbol{\sigma} \times \hat{\mathbf{P}})_z + \frac{\mu_a H}{m_0 c^2}$$

in a homogeneous magnetic field is taken as a spin integral of motion characterizing the electron-spin polarization relative to the magnetic field vector. Then we have

$$\hat{H} \psi = E_n \zeta \psi, \quad \hat{P}_z \psi = p_z \psi, \quad \hat{J}_z \psi = \hbar \left(l - \frac{1}{2} \right) \psi, \quad (\text{B1a})$$

$$\hat{\Pi}_z \psi = \zeta b \psi, \quad \zeta = \pm 1. \quad (\text{B1b})$$

In cylindrical coordinates (r, φ, z) , the wave function that is a solution to the Dirac-Pauli equation and satisfies Eqs. (B1) has the form [7]

$$\psi(\mathbf{r}, t) = \frac{\sqrt{e_0 H}}{\sqrt{L} \sqrt{2\pi\hbar c}} e^{-iE_n \zeta t/\hbar} e^{ip_z z/\hbar} e^{i(l-1)\varphi} f(\rho, \varphi), \quad (\text{B2})$$

where

$$f(\rho, \varphi) = \begin{pmatrix} C_1 I_{n-1,s}(\rho) \\ iC_2 I_{n,s}(\rho) e^{i\varphi} \\ C_3 I_{n-1,s}(\rho) \\ iC_4 I_{n,s}(\rho) e^{i\varphi} \end{pmatrix}.$$

Here $n = l + s = 0, 1, 2, \dots$ is the principal quantum number, $s = 0, 1, 2, \dots$ is the radial quantum number, and $l = 0, \pm 1, \pm 2, \dots$ is the azimuthal quantum number. In dimensionless form, the spin coefficients C_i are determined as

$$\begin{aligned} C_1 &= \frac{\zeta}{2} \sqrt{\frac{1}{2} \left(1 + \zeta \frac{1}{b} \right)} \left[\sqrt{1 + \frac{b_z}{B_{n\zeta}}} + \zeta \sqrt{1 - \frac{b_z}{B_{n\zeta}}} \right], \\ C_2 &= \frac{\zeta}{2} \sqrt{\frac{1}{2} \left(1 - \zeta \frac{1}{b} \right)} \left[\sqrt{1 - \frac{b_z}{B_{n\zeta}}} - \zeta \sqrt{1 + \frac{b_z}{B_{n\zeta}}} \right], \\ C_3 &= \frac{\zeta}{2} \sqrt{\frac{1}{2} \left(1 + \zeta \frac{1}{b} \right)} \left[\sqrt{1 + \frac{b_z}{B_{n\zeta}}} - \zeta \sqrt{1 - \frac{b_z}{B_{n\zeta}}} \right], \\ C_4 &= \frac{1}{2} \sqrt{\frac{1}{2} \left(1 - \zeta \frac{1}{b} \right)} \left[\sqrt{1 + \frac{b_z}{B_{n\zeta}}} + \zeta \sqrt{1 - \frac{b_z}{B_{n\zeta}}} \right], \end{aligned} \quad (\text{B3})$$

where the electron energy is given as

$$B_{n\zeta} = \frac{E_n \zeta}{m_0 c^2} = \sqrt{b_z^2 + \left(\sqrt{1 + b_\perp^2} + \zeta \frac{\mu_a H}{m_0 c^2} \right)^2}, \quad (\text{B4})$$

$$b_z = \frac{p_z}{m_0 c}, \quad b_\perp = 2 \sqrt{\frac{\mu_0 H}{m_0 c^2} n}, \quad b = \sqrt{1 + b_\perp^2}.$$

Let us give the exact forms of all necessary matrix elements of the momentum and spin operators:

$$\langle \psi_{m'\zeta'} | \hat{P}_x | \psi_{m\zeta} \rangle = \frac{i}{2} m_0 c b_{\perp} (\delta_{m'-m,+1} - \delta_{m'-m,-1}) \delta_{\zeta'\zeta},$$

$$\langle \psi_{m'\zeta'} | \hat{P}_y | \psi_{m\zeta} \rangle = \frac{1}{2} m_0 c b_{\perp} (\delta_{m'-m,+1} + \delta_{m'-m,-1}) \delta_{\zeta'\zeta},$$

$$\langle \psi_{m'\zeta'} | \hat{P}_z | \psi_{m\zeta} \rangle = m_0 c b_z \delta_{m'm} \delta_{\zeta'\zeta},$$

$$\langle \psi_{m'\zeta'} | \hat{S}_x | \psi_{m\zeta} \rangle = \frac{i}{2} [b - \zeta (\delta_{m'-m,+1} - \delta_{m'-m,-1})] \\ \times (\delta_{m'-m,+1} - \delta_{m'-m,-1}) \delta_{-\zeta'\zeta},$$

$$\langle \psi_{m'\zeta'} | \hat{S}_y | \psi_{m\zeta} \rangle = \frac{1}{2} [b - \zeta (\delta_{m'-m,+1} - \delta_{m'-m,-1})] \\ \times (\delta_{m'-m,+1} + \delta_{m'-m,-1}) \delta_{-\zeta'\zeta},$$

$$\langle \psi_{m'\zeta'} | \hat{S}_z | \psi_{m\zeta} \rangle = \left(\zeta \frac{B_{n\zeta}}{b} \delta_{\zeta'\zeta} + \frac{b_{\perp} b_z}{b} \delta_{-\zeta'\zeta} \right) \delta_{m'm},$$

$$\langle \psi_{m'\zeta'} | \hat{S}^0 | \psi_{m\zeta} \rangle = \left(\zeta \frac{b_z}{b} \delta_{\zeta'\zeta} + \frac{B_{n\zeta} b_{\perp}}{b} \delta_{-\zeta'\zeta} \right) \delta_{m'm}.$$

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